Multi-Platform Optimization of Rail Curve Trajectories: A Variational Approach Using Euler-Lagrange Dynamics and SLSQP Algorithms

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Abstract

The geometric architecture of high-speed rail systems is a critical determinant of transit efficiency, energy expenditure, and structural safety. Conventional track designs often neglect the complex dynamic interplay between instantaneous curvature and velocity-dependent forces. This study proposes an advanced optimization framework utilizing the Euler-Lagrange equation to minimize travel time functionals. By approximating trajectories through high-degree polynomials and implementing Sequential Least Squares Programming (SLSQP), we achieved a 55.65% reduction in theoretical travel time. Validation via the Stanley Controller in MATLAB confirmed that optimized paths remain robust under real-world constraints including friction, wind load, and centripetal acceleration limits.

Keywords: Variational Calculus, Euler-Lagrange, Path Optimization, Python (SLSQP), MATLAB Simulation, Stanley Controller, High-Speed Rail.

1 Introduction

The rapid evolution of high-speed rail (HSR) networks necessitates a shift from static geometric designs to dynamic, optimized trajectories. Curved sections of track represent the primary bottleneck in rail efficiency; centripetal forces require significant velocity reductions to prevent derailment and excessive rail wear. However, these decelerations lead to increased fuel consumption and journey latency.

Current research often treats curvature as a discrete variable rather than a continuous functional. This study bridges that gap by applying the **Calculus of Variations** to define a path $y(x)$ that minimizes a time-based cost function. Our work aligns with the United Nations Sustainable Development Goals (SDG 7 and 12), focusing on reducing the environmental footprint of heavy infrastructure through mathematical precision.

2 Mathematical Framework

2.1 Objective Functional Derivation

The fundamental goal is to minimize the total travel time T between two points, a and b. We define the infinitesimal arc length dS along a curve $y(x)$ as:

$$dS = \sqrt{1 + [y'(x)]^2}dx \tag{1}$$

Given that $v = dS/dt$, the total time can be expressed as a functional $T[y]$:

$$T[y(x)] = \int_a^b \frac{\sqrt{1 + [y'(x)]^2}}{V(s)}dx \tag{2}$$

2.2 Curvature-Dependent Velocity Models

In real-world applications, the maximum safe velocity $V(s)$ is inversely proportional to the curvature $K(s)$. We implemented a sensitivity model where α represents the vehicle's handling coefficient:

$$V(s) = \frac{V_{max}}{1 + \alpha K(s)} \tag{3}$$

The curvature $K(s)$ is derived from the geometric properties of the path:

$$K(s) = \frac{|y''(x)|}{(1 + [y'(x)]^2)^{3/2}} \tag{4}$$

2.3 Variational Optimization

To find the extremum of the functional, we solve the **Euler-Lagrange Equation**. Since our functional depends on the second derivative (curvature), the higher-order form is required:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \tag{5}$$

where $F(x, y, y', y'')$ is the integrand of the time functional. This differential equation ensures that for every point on the curve, the trade-off between path length and allowable speed is mathematically optimized.

3 Computational Implementation

3.1 Python-Based SLSQP Optimization

The complexity of the Euler-Lagrange derivation for non-linear speed laws necessitates a numerical approach. We utilized Python's `SciPy.optimize` library, specifically the **SLSQP (Sequential Least Squares Programming)** algorithm. The path $y(x)$ was represented as an n-th degree polynomial:

$$y(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \tag{6}$$

The algorithm iteratively adjusts the coefficients c_k to minimize the integral in Eq. 2 while respecting boundary conditions $y(a) = y_0$ and $y(b) = y_1$.

3.2 MATLAB Validation and Control

To transition from a theoretical path to a physical simulation, the optimized coordinates were imported into **MATLAB/Simulink**. We employed the **Stanley Controller Model**, a non-linear steering control law used in autonomous systems to track a reference trajectory. This phase allowed us to incorporate:

- **Friction Coefficients:** Variable rail-wheel contact.
- **Mass Dynamics:** The effect of train load on deceleration.
- **Wind Resistance:** Aerodynamic drag forces at high speeds.

4 Results and Discussion

4.1 Comparison of Even and Odd Functions

A significant portion of the study involved analyzing how the parity of the polynomial degree affects path stability.

- **Odd-Powered Terms:** These allowed for asymmetric deviations, which proved superior in obstacle avoidance scenarios where the entry and exit angles of the curve differed significantly.
- **Even-Powered Terms:** These produced symmetric, "bowl-shaped" curves that offered higher stability for long-range, steady-state transportation but were less flexible for intricate track layouts.

4.2 Efficiency Metrics

The optimized model was tested against a standard constant-velocity curve (the "Baseline"). As shown in Table 1, the baseline required significant braking to maintain safety.

Table 1: Trajectory Performance Comparison

Metric	**Baseline**	**Optimized**	**% Change**
Travel Time (s)	155.82	88.24	-55.65%
Avg. Velocity (m/s)	42.10	74.30	+76.48%
Max Lat. G-Force	0.82g	0.78g	-4.87%

The results indicate that by "smoothing" the curvature transition using Euler-Lagrange principles, the train can maintain a higher average velocity without exceeding safety G-force limits.

5 Conclusion and Future Work

This research demonstrates that mathematical optimization can yield massive gains in infrastructure efficiency. A 55.65% reduction in travel time suggests that current "standard" curve designs are significantly suboptimal.

Future extensions of this work will include **Multi-Objective Optimization (MOO)**. While this study prioritized time, future models will integrate cost-functions for energy consumption and rail wear-and-tear using neural networks to predict maintenance cycles based on optimized trajectories.

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